REVERSE MATHEMATICS AND INITIAL INTERVALS

EMANUELE FRITTAION AND ALBERTO MARCONE

ABSTRACT. In this paper we study the reverse mathematics of two theorems by Bonnet about partial orders. These results concern the structure and cardinality of the collection of the initial intervals. The first theorem states that a partial order has no infinite antichains if and only if its initial intervals are finite unions of ideals. The second one asserts that a countable partial order is scattered and does not contain infinite antichains if and only if it has countably many initial intervals. We show that the left to right directions of these theorems are equivalent to ACA_0 and ATR_0, respectively. On the other hand, the opposite directions are both provable in WKL_0, but not in RCA_0. We also prove the equivalence with ACA_0 of the following result of Erdös and Tarski: a partial order with no infinite strong antichains has no arbitrarily large finite strong antichains.

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1. Introduction

In this paper we study from the viewpoint of reverse mathematics some theorems dealing with the structure and the cardinality of the collection of initial intervals (also called downward closed subsets) in a partial order. Recall that an ideal is an initial interval such that every pair of elements is compatible (i.e. has a common upper bound) in the interval.

The first result is a characterization of partial orders with no infinite antichains in terms of the decomposition of initial intervals into union of ideals. It is due to Bonnet [Bon75, Lemma 2] and can be found in Fraïssé's monograph [Fra00, §4.7.2]:

Theorem 1.1. *A partial order has no infinite antichains if and only if every initial interval is a finite union of ideals.*

In [PS06] Theorem 1.1 is attributed to Erdös and Tarski because its 'hard' (left to right) direction can be deduced quite easily from the following result, which is part of [ET43, Theorem 1]:

Theorem 1.2. *If a partial order has no infinite strong antichains then it has no arbitrarily large finite strong antichains.*

Here, by strong antichain we mean a set of pairwise incompatible (and not only incomparable, as in antichain) elements. (Notice that Erdös and Tarski work with what we would call filters and final intervals.)

An intermediate step between Theorems 1.2 and 1.1 is the following characterization of partial orders with no infinite strong antichains:

Theorem 1.3. *A partial order has no infinite strong antichains if and only if it is a finite union of ideals.*

Our proof of Lemma 4.2 shows how to deduce the left to right direction of Theorem 1.3 from Theorem 1.2.

In [Bon75] Theorem 1.1 is a step in the proof of the following result, which is also featured in Fraïssé's monograph [Fra00, §6.7]:

Theorem 1.4. *If an infinite partial order P is scattered (i.e. there is no embedding of the rationals into P) and has no infinite antichains, then the set of initial intervals of P has the same cardinality of P.*

The converse of Theorem 1.4 is in general false, but it holds when $|P| < 2^{\aleph_0}$, and in particular when P is countable:

Theorem 1.5. *A countable partial order is scattered and has no infinite antichains if and only if it has countably many initial intervals.*

The program of reverse mathematics ([Sim09] is the basic reference) gauges the strength of mathematical theorems by means of the subsystems of second order arithmetic necessary for their proofs. This approach allows only the study of statements about countable (or countably coded) objects. We therefore study the strength of Theorem 1.5 and of the restrictions of Theorems 1.1, 1.2 and 1.3 to countable partial orders. We notice that [ET43, Bon75, Fra00] put no restriction on the cardinality of the partial order and therefore often use set-theoretic techniques which are not available in (subsystems of) second order arithmetic. On the other hand we can always assume that the partial orders are defined on a subset of the set of the natural numbers, and this is on occasion helpful.

Since Theorems 1.1, 1.3, and 1.5 are equivalences, we study separately the two implications, which turn out to have different axiomatic strengths. In particular, the 'easy' (right to left) directions of Theorems 1.1 and 1.5 are quite interesting from the viewpoint of reverse mathematics and we are not able to settle the problem of establishing their strength, leaving open the possibility that they have strength intermediate between RCA_0 and WKL_0.

We assume familiarity with the 'big five' of reverse mathematics, namely, in order of increasing strength, RCA_0, WKL_0, ACA_0, ATR_0, and $\mathbf{\Pi}^1_1\text{-}\mathsf{CA}_0$.

We now state our main results and at the same time describe the organization of the paper. In section 2 we establish our notation and terminology and recall some basic results. In section 3 we prove a couple of technical lemmas that are useful later on.

In Section 4 we consider Theorem 1.2 and the left to right directions of Theorems 1.1, 1.3, and 1.5. Subsection 4.1 culminates in Theorem 4.5 where we prove, over RCA_0, the equivalence of ACA_0 with each of the three statements:

- in a countable partial order with no infinite antichains every initial interval is a finite union of ideals;
- in a countable partial order with no infinite strong antichains there is a bound on the size of the strong antichains;
- every countable partial order with no infinite strong antichains is a finite union of ideals.

In subsection 4.2 we show that the statement

- every countable partial order which is scattered and has no infinite antichains has countably many initial intervals.

is equivalent to ATR_0 over ACA_0 (Theorem 4.8). To obtain the reversal we slightly modify a proof in [Clo89].

In section 5 we deal with the right to left directions of Theorems 1.1, 1.3, and 1.5, i.e. with the statements:

- if every initial interval of a countable partial order is a finite union of ideals, then the partial order has no infinite antichains;
- if a countable partial order is a finite union of ideals then it has no infinite strong antichains;
- if a countable partial order has countably many initial intervals, then it has no infinite antichains;
- if a countable partial order has countably many initial intervals, then it is scattered.

The obvious proofs of these statements go through in ACA_0, but we show that they are all provable in weaker systems. In fact RCA_0 proves the second and fourth statement (Lemma 5.1 and Theorem 5.2). On the other hand, the first and third statement are both provable in WKL_0 (Theorems 5.6 and 5.10) and fail in the ω-model of computable sets and hence cannot be proved in RCA_0 (Theorems 5.12 and 5.13). Our results thus do not completely determine the strength of these two statements.

In Section 6 we briefly discuss the open problems left by our results and mention some partial answers obtained by other authors after a first draft of this paper was circulated.

2. Terminology, notation, and basic facts

All definitions in this section are made in RCA_0.

2.1. **Finite sequences and trees.** We typically use σ and τ to denote finite sequences of natural numbers, that is elements of $\mathbb{N}^{<\mathbb{N}}$. Often they belong to $2^{<\mathbb{N}}$, i.e. they are binary, and in one occasion to $3^{<\mathbb{N}}$, i.e. they are ternary. Let $|\sigma|$ be the length of σ and list it as $\langle \sigma(0), \dots, \sigma(|\sigma|-1) \rangle$. In particular $\langle \rangle$ is the unique sequence of length 0. We write $\sigma \sqsubseteq \tau$ to mean that σ is an initial segment of τ, while $\sigma^\frown \tau$ denotes the concatenation of σ and τ. By $\sigma \restriction k$ we mean the initial segment of σ of length k and similarly, when f is a function, $f \restriction k$ is the finite sequence $\langle f(0), \dots, f(k-1) \rangle$.

A *tree* T is a set of finite sequences such that $\tau \in T$ and $\sigma \sqsubseteq \tau$ imply $\sigma \in T$. A tree is *pruned* if it contains no endnodes, i.e. $(\forall \sigma \in T)(\exists \tau \in T)\sigma \sqsubset \tau$. A *path* in T is a function f such that for all n the finite sequence $f \restriction n$ belongs to T. We write $[T]$ to denote the collection of all paths in T: $[T]$ does not formally exists in second order arithmetic, but $f \in [T]$ is a convenient shorthand.

A tree T is *perfect* if for all $\sigma \in T$ there exist $\tau_0, \tau_1 \in T$ such that $\sigma \sqsubseteq \tau_0, \tau_1$ and neither $\tau_0 \sqsubseteq \tau_1$ nor $\tau_1 \sqsubseteq \tau_0$ hold. A tree T has *countably many paths* if there exists a sequence $\{f_n : n \in \mathbb{N}\}$ (coded by a single set) such that for every $f \in [T]$ we have $f = f_n$ for some $n \in \mathbb{N}$.

If T does not have countably many paths then we say that it has *uncountably many paths*.

By [Sim09, Theorem V.5.5] ATR_0 is equivalent to the perfect tree theorem:

Theorem 2.1 (ACA_0). *The following are equivalent:*

(1) ATR_0;
(2) *every tree with uncountably many paths contains a perfect subtree.*

2.2. **Partial orders.** Within RCA_0 saying that $(P, \preceq)$ is a partial order means that $P \subseteq \mathbb{N}$ and $\preceq \, \subseteq P \times P$ is reflexive, antisymmetric and transitive. As usual, we use $\prec$ to denote the strict order. From now on we refer to $(P, \preceq)$ simply as P. When we deal with several partial orders at the same time, we use subscripts as in $\preceq_P$ to distinguish between the relations.

Finite partial orders can easily be studied in RCA_0 and hence, whenever it is convenient and without further notice, we assume that P is infinite.

Every time we define a partial order $\preceq$ on a set P we assume reflexivity, and focus on explaining when distinct elements are related and on checking transitivity.

We say that $x, y \in P$ are *comparable* if $x \preceq y$ or $y \preceq x$. If x and y are incomparable we write $x \perp y$. A partial order P is a *linear order* if all its elements are pairwise comparable. A linear order P is *dense* if for all $x, y \in P$ such that $x \prec y$ there exists $z \in P$ with $x \prec z \prec y$.

A subset $D \subseteq P$ is an *antichain* if all its elements are pairwise incomparable, i.e.

$$(\forall x, y \in D)(x \neq y \implies x \perp y).$$

We say that $x, y \in P$ are *compatible in* P if there is $z \in P$ such that $x \preceq z$ and $y \preceq z$. Notice that two elements of P might be compatible in P but not in some $X \subseteq P$ to which they belong.

A subset $S \subseteq P$ is a *strong antichain in* P if its elements are pairwise incompatible in P, i.e.

$$(\forall x, y \in S)(\forall z \in P)(x, y \preceq z \implies x = y).$$

A subset $I \subseteq P$ is an *initial interval* of P if

$$(\forall x, y \in P)(x \preceq y \wedge y \in I \implies x \in I).$$

An initial interval A of P is an *ideal* if every two elements of A are compatible in A, i.e.

$$(\forall x, y \in A)(\exists z \in A)(x \preceq z \wedge y \preceq z).$$

If $x \in P$ we let $P_{\perp x} = \{y \in P \colon x \perp y\}$ and define the *upper and lower cones* determined by x setting

$$P_{\succeq x} = \{y \in P \colon x \preceq y\} \text{ and } P_{\preceq x} = \{y \in P \colon y \preceq x\}.$$

$P_{\succ x}$ and $P_{\prec x}$ are defined in the obvious way. If $X \subseteq P$ we write $\downarrow X$ for the *downward closure* of X, i.e. $\bigcup_{x \in X} P_{\preceq x}$. Notice that the existence of $\downarrow X$ as a set is equivalent to ACA_0 over RCA_0.

2.3. **Well-partial orders, scattered partial orders and lexicographic sums.** A partial order P is *well-founded* if P contains no infinite descending sequence, i.e. no function $f \colon \mathbb{N} \to P$ such that $f(i) \succ f(j)$ for all $i < j$. A well-founded linear order is a *well-order*.

A partial order P is a *well-partial order* if for every function $f \colon \mathbb{N} \to P$ there exist $i < j$ such that $f(i) \preceq f(j)$. There are many classically equivalent definitions of well-partial order. In particular a well-partial order is a well-founded partial order with no infinite antichains. For a reverse mathematics study of these equivalences we refer to [CMS04]. For our purposes, it is enough to know that all these equivalences are provable in ACA_0 and that RCA_0 suffices to show that every well-partial order is well-founded and has no infinite antichains.

The *Kleene-Brouwer order* on finite sequences is the linear order defined by $\sigma \leq_{\mathrm{KB}} \tau$ if either $\tau \sqsubseteq \sigma$ or $\sigma(i) < \tau(i)$ for the least i such that $\sigma(i) \neq \tau(i)$. One of the main features of $\leq_{\mathrm{KB}}$ is that, provably in ACA_0, its restriction to a tree T is a well-order if and only if T has no paths ([Sim09, Lemma V.1.3]).

An *embedding* of a partial order Q into a partial order P is a function $f \colon Q \to P$ such that for all $x, y \in Q$ we have $x \preceq_Q y$ if and only if $f(x) \preceq_P f(y)$. A partial order P is *scattered* if there is no embedding of $\mathbb{Q}$ (the order of the rationals) into P.

Lemma 2.2 (RCA_0). *A partial order is scattered if and only if it does not contain any dense linear order.*

Proof. The left to right is immediate because RCA_0 suffices to carry out the usual back-and-forth argument. For the other direction, given an embedding $f : \mathbb{Q} \to P$ by recursion we can find $D \subseteq \mathbb{Q}$ dense such that f restricted to D is strictly increasing with respect to the ordering of the natural numbers. Thus the range of f restricted to D exists in RCA_0 and is a dense linear order. $\square$

If P is a partial order and $\{P_x \colon x \in P\}$ is a sequence of partial orders indexed by P we define the lexicographic sum of the P_x along P, denoted by $\sum_{x \in P} P_x$, to be the partial order on the set $Q = \{(x, y) \colon x \in P \wedge y \in P_x\}$ defined by

$$(x, y) \preceq_Q (x', y') \iff x \prec_P x' \vee (x = x' \wedge y \preceq_{P_x} y').$$

Lemma 2.3 (RCA_0). *The lexicographic sum of scattered partial orders along a scattered partial order is scattered.*

Proof. Let $Q = \sum_{x \in P} P_x$ be a lexicographic sum and suppose that Q is not scattered. Fix an embedding $f \colon \mathbb{Q} \to Q$.

First suppose that for some $a <_{\mathbb{Q}} b$ and $x \in P$ we have $f(a) = (x, y)$ and $f(b) = (x, y')$. Then, the composition of f with the projection on the second coordinate is an embedding of the rational interval $(a, b)_{\mathbb{Q}}$ into P_x. Since $\mathbb{Q}$ embeds into its open intervals, P_x is not scattered.

Otherwise, composing f with the projection on the first coordinate, we obtain an embedding of $\mathbb{Q}$ into P, and P is not scattered. $\qquad\square$

2.4. **The set of initial intervals.** We denote by $\mathcal{I}(P)$ the collection of initial intervals of the partial order P. In second order arithmetic, $\mathcal{I}(P)$ does not formally exist, and $I \in \mathcal{I}(P)$ is a shorthand for the formula "I is an initial interval of P". To study Theorem 1.5 we need to discuss the cardinality of $\mathcal{I}(P)$.

We say that the partial order P has *countably many initial intervals* if there exists a sequence $\{I_n : n \in \mathbb{N}\}$ such that for every $I \in \mathcal{I}(P)$ we have $I = I_n$ for some $n \in \mathbb{N}$. Otherwise, we say that P has *uncountably many initial intervals*.

Within ACA_0 we can prove that, if P has countably many initial intervals, then there exists a sequence $\{I_n : n \in \mathbb{N}\}$ such that $I \in \mathcal{I}(P)$ if and only if there exists $n \in \mathbb{N}$ such that $I = I_n$. In this case we write $\mathcal{I}(P) = \{I_n : n \in \mathbb{N}\}$.

The partial order P has *perfectly many initial intervals* if there exists a nonempty perfect tree $T \subseteq 2^{<\mathbb{N}}$ such that $[T] \subseteq \mathcal{I}(P)$, that is, for all $f \in [T]$, the set $\{x \in \mathbb{N} : f(x) = 1\} \in \mathcal{I}(P)$.

A useful tool for studying the notions we just defined is the *tree of finite approximations of initial intervals* of the partial order P. We define the tree $T(P) \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in T(P)$ if and only if for all $x, y < |\sigma|$:

- $\sigma(x) = 1$ implies $x \in P$;
- $\sigma(y) = 1$ and $x \preceq y$ imply $\sigma(x) = 1$.

Notice that $T(P)$ is a pruned tree and that the paths in $T(P)$ are exactly the characteristic functions of the initial intervals of P. From the latter observation we easily obtain:

Lemma 2.4 (RCA_0). *Let P be a partial order.*

(i) *P has countably many initial intervals if and only if $T(P)$ has countably many paths;*
(ii) *P has perfectly many initial intervals if and only if $T(P)$ contains a perfect subtree.*

In particular, the formula "P has perfectly many initial intervals" is provably $\mathbf{\Sigma}_1^1$ within RCA_0. Moreover a straightforward diagonal argument shows in RCA_0 that a nonempty perfect tree has uncountably many paths. Therefore we have that RCA_0 proves that a partial order with perfectly many initial intervals has uncountably many initial intervals. Using the perfect tree theorem we obtain that ATR_0 proves

that a partial order with uncountably many initial intervals has actually perfectly many initial intervals. This implies that the formula "P has uncountably many initial intervals" is provably $\mathbf{\Sigma}_1^1$ within ATR_0.

In connection with this recall the following result due to Peter Clote [Clo89]:

Theorem 2.5 (ACA_0). *The following are equivalent:*

(1) ATR_0*;*
(2) *any linear order has countably many or perfectly many initial intervals;*
(3) *any scattered linear order has countably many initial intervals.*

Clote actually states the equivalence of ATR_0 only with (2), but his proofs yield also the equivalence with (3).

2.5. **The system** ATR_0^X. Recall that, by [Sim09, Theorem VIII.3.15], ATR_0 is equivalent over ACA_0 to the statement

$$(\forall X)(\forall a \in \mathcal{O}^X)(H_a^X \text{ exists})$$

where $\mathcal{O}^X$ is the collection of (indices for) X-computable ordinals and H_a^X codes the iteration of the jump along a starting from X. This naturally leads to consider lightface versions of ATR_0, as in [Tan89], [Tan90], and [Mar91]. Here we make explicit mention of the set parameter we use (rather then deal only with the parameterless case and then invoke relativization) and let ATR_0^X be ACA_0 plus the formula $(\forall a \in \mathcal{O}^X)(H_a^X \text{ exists})$. In ATR_0^X one can prove arithmetical transfinite recursion along any X-computable well-order.

By checking the proof of the forward direction of Theorem 2.1 one readily realizes that ATR_0^X proves the perfect tree theorem for X-computable trees:

Theorem 2.6 (ATR_0^X). *Every X-computable tree with uncountably many paths contains a perfect subtree.*

The following is [Sim09, Lemma VIII.4.19]:

Theorem 2.7 (ATR_0^X). *There exists a countable coded ω-model M such that $X \in M$ and M satisfies $\mathbf{\Sigma}_1^1\text{-}\mathsf{DC}_0$.*

We will use the following corollary:

Corollary 2.8 (ATR_0). *For all X and Y there exists a countable coded ω-model M such that $X, Y \in M$ and M satisfies both $\mathbf{\Sigma}_1^1\text{-}\mathsf{DC}_0$ and ATR_0^X.*

Proof. We argue in ATR_0 and let X and Y be given. By $\mathbf{\Sigma}_1^1\text{-}\mathsf{AC}_0$, which is a consequence of ATR_0, the main axiom of ATR_0^X is equivalent to a $\mathbf{\Sigma}_1^1$ formula $(\exists Z)\varphi(Z, X)$ with φ arithmetic. This formula is true in ATR_0, and hence we can fix Z such that $\varphi(Z, X)$. By Theorem 2.7 there exists a countable coded ω-model M of $\mathbf{\Sigma}_1^1\text{-}\mathsf{DC}_0$ such that

$X \oplus Y \oplus Z \in M$. In particular, $X, Y \in M$ and, as $Z \in M$ and M is a model of $\Sigma_1^1\text{-DC}_0$ (hence also of $\Sigma_1^1\text{-AC}_0$), M satisfies ATR_0^X. $\qquad\square$

3. Initial interval separation and essential unions

In this section we prove two technical results that are useful in the remainder of the paper.

3.1. **Initial interval separation.** Our first result is a new equivalence with WKL_0, inspired by the usual Σ_1^0 separation ([Sim09, Lemma IV.4.4]) but producing separating sets which are also initial intervals.

Lemma 3.1. *Over RCA_0, the following are equivalent:*

(1) WKL_0;

(2) Σ_1^0 *initial interval separation. Let P be a partial order and $\varphi(x)$, $\psi(x)$ be Σ_1^0 formulas with one distinguished free number variable.*

If $(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \npreceq x)$, then there exists an initial interval I of P such that

$$(\forall x \in P)((\varphi(x) \implies x \in I) \wedge (\psi(x) \implies x \notin I)).$$

(3) *Initial interval separation. Let P be a partial order and suppose $A, B \subseteq P$ are such that $(\forall x \in A)(\forall y \in B)y \npreceq x$. Then there exists an initial interval I of P such that $A \subseteq I$ and $B \cap I = \emptyset$.*

Proof. We first assume WKL_0 and prove (2). Fix the partial order P and let $\varphi(x) \equiv (\exists m)\varphi_0(x, m)$ and $\psi(n) \equiv (\exists m)\psi_0(x, m)$ be Σ_1^0 formulas with φ_0 and ψ_0 Σ_0^0. Assume $(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \npreceq x)$.

Form the binary tree $T \subseteq 2^{<\mathbb{N}}$ by letting $\sigma \in T$ if and only if $\sigma \in T(P)$ and for all $x, y < |\sigma|$:

(i) $(\exists m < |\sigma|)\, \varphi_0(x, m) \implies \sigma(x) = 1$, and

(ii) $(\exists m < |\sigma|)\, \psi_0(x, m) \implies \sigma(x) = 0$.

To see that T is infinite, we show that for every $k \in \mathbb{N}$ there exists $\sigma \in T$ with $|\sigma| = k$. Given k let

$$\sigma(x) = 1 \iff x \in P \wedge (\exists y, m < k)(\varphi_0(y, m) \wedge x \preceq y)$$

for all $x < k$. It is easy to verify that $\sigma \in T$. By weak König's lemma, T has a path f. By Σ_0^0 comprehension, let $I = \{x \colon f(x) = 1\}$. It is straightforward to see that I is as desired.

(3) is the special case of (2) obtained by considering the Σ_0^0, and hence Σ_1^0, formulas $x \in A$ and $x \in B$.

It remains to prove (3) $\implies$ (1). It suffices to derive in RCA_0 from (3) the existence of a set separating the disjoint ranges of two one-to-one functions ([Sim09, Lemma IV.4.4]). Let $f, g \colon \mathbb{N} \to \mathbb{N}$ be one-to-one functions such that $(\forall n, m \in \mathbb{N})f(n) \neq g(m)$. Define a partial order on $P = \{a_n, b_n, c_n \colon n \in \mathbb{N}\}$ by letting $c_n \preceq a_m$ if and only if $f(m) = n$, $b_m \preceq c_n$ if and only if $g(m) = n$, and adding no other

comparabilities. Let $A = \{a_n \colon n \in \mathbb{N}\}$ and $B = \{b_n \colon n \in \mathbb{N}\}$, so that $(\forall x \in A)(\forall y \in B) y \not\leq x$. By (3) there exists an initial interval I of P such that $A \subseteq I$ and $B \cap I = \emptyset$. It is easy to check that $\{n \colon c_n \in I\}$ separates the range of f from the range of g. □

3.2. **Essential unions of sets.** Our second result deals with finite unions of sets and will be applied to finite unions of ideals.

Definition 3.2 (RCA$_0$). Let $I \subseteq \mathbb{N}$. A family of sets $\{A_i \colon i \in I\}$ is *essential* if
$$(\forall i \in I)(A_i \not\subseteq \bigcup_{j \in I, j \neq i} A_j).$$
The union of such a family is called an *essential union.*

Not every family of sets can be made essential without loosing elements from the union. The simplest example is a sequence $\{A_n \colon n \in \mathbb{N}\}$ of sets such that $A_n \subset A_{n+1}$ for every n. However the following shows that, provably in RCA$_0$, every finite family of sets can be made essential.

Lemma 3.3 (RCA$_0$). *For every family of sets $\{A_i \colon i \in F\}$ with F finite there exists $I \subseteq F$ such that $\{A_i \colon i \in I\}$ is essential and*
$$\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i.$$

Proof. Let
$$n_0 = \min\{n \colon (\exists I \subseteq F)(|I| = n \wedge \bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i)\}.$$

RCA$_0$ proves that n_0 exists, otherwise by $\mathbf{\Sigma}_1^0$-induction one could prove
$$(\forall n)(\forall I \subseteq F)(|I| \leq n \to \bigcup_{i \in F} A_i \neq \bigcup_{i \in I} A_i),$$
which is clearly false.

If $I \subseteq F$ is such that $|I| = n_0$ and $\bigcup_{i \in F} A_i = \bigcup_{i \in I} A_i$ then it is immediate that $\{A_i \colon i \in I\}$ is essential. □

4. The left to right directions

In this section we study Theorem 1.2 and the left to right directions of Theorems 1.1, 1.3, and 1.5. It turns out that the left to right direction of Theorem 1.5 is equivalent to ATR$_0$ and the other statements are equivalent to ACA$_0$.

4.1. **Equivalences with ACA$_0$.** We consider the following equivalence, which includes Theorems 1.2 and 1.3.

Theorem 4.1. *Let P be a partial order. Then the following are equivalent:*

   (1) P is a finite union of ideals;

(2) there is a finite bound on the size of the strong antichains in P;
(3) P has no infinite strong antichains.

We notice that (1) $\implies$ (2) and (2) $\implies$ (3) are easily provable in RCA_0. We show that (2) $\implies$ (1) and (3) $\implies$ (2) are provable in ACA_0.

We start with implication (2) $\implies$ (1).

Lemma 4.2 (ACA_0). *Let P be a partial order with no arbitrarily large finite strong antichains. Then P is a finite union of ideals.*

Proof. Let $\ell \in \mathbb{N}$ be the maximum size of a strong antichain in P and let S be a strong antichain of size ℓ. For every $z \in S$ define by arithmetical comprehension

$$A_z = \{x \in P \colon x \text{ and } z \text{ are compatible}\}.$$

Since S is maximal with respect to inclusion it is immediate that $P = \bigcup_{z \in S} A_z$ and it suffices to show that each A_z is an ideal.

Fix $z \in S$ and $x, y \in A_z$. Let x_0, y_0 be such that $x \preceq x_0$, $y \preceq y_0$, and $z \preceq x_0, y_0$. It suffices to show that x_0 and y_0 are compatible in A_z. If this is not the case, x_0 and y_0 are incompatible also in P (because $P_{\succeq x_0} \subseteq P_{\succeq z} \subseteq A_z$). Moreover for each $w \in S \setminus \{z\}$ each of x_0 and y_0 is incompatible with w in P because z and w are incompatible in P. Thus $(S \setminus \{z\}) \cup \{x_0, y_0\}$ is a strong antichain of size $\ell + 1$, a contradiction. $\square$

To obtain (3) $\implies$ (2) of Theorem 4.1 we are going to use the existence of maximal (with respect to inclusion) strong antichains. We first show that this statement is equivalent to ACA_0.

Lemma 4.3. *Over RCA_0, the following are equivalent:*

(1) ACA_0;
(2) every strong antichain in a partial order extends to a maximal strong antichain;
(3) every partial order contains a maximal strong antichain.

Proof. We show (1) $\implies$ (2). Let P be a partial order and $S \subseteq P$ be a strong antichain. By recursion we dene $f \colon \mathbb{N} \to \{0, 1\}$ by letting $f(x) = 1$ if and only if $S \cup \{y < x \colon f(y) = 1\} \cup \{x\}$ is a strong antichain in P. Then $T = \{x \colon f(x) = 1\}$ is a maximal strong antichain with $S \subseteq T$.

Implication (2) $\implies$ (3) is trivial. To show (3) $\implies$ (1), we argue in RCA_0 and derive from (3) the existence of the range of any one-to-one function. Given $f \colon \mathbb{N} \to \mathbb{N}$ one-to-one consider $P = \{a_n, b_n, c_n \colon n \in \mathbb{N}\}$. For all $n, m \in \mathbb{N}$ let $a_n \preceq c_m$ if and only if $b_n \preceq c_m$ if and only if $f(m) = n$, and add no other comparabilities. By (3), let $S \subseteq P$ be a maximal strong antichain. Then, n belongs to the range of f if and only if $a_n \notin S \vee b_n \notin S$. Thus the range of f exists by Σ_0^0 comprehension. $\square$

The following is implication (3) $\implies$ (2) of Theorem 4.1, i.e. our formalization of the left to right direction of Theorem 1.2.

Lemma 4.4 (ACA_0). *Let P be a partial order with no infinite strong antichains. Then there are no arbitrarily large finite strong antichains in P.*

Proof. Suppose for a contradiction that P has arbitrarily large finite strong antichains but no infinite strong antichains (the existence of such a pair is proved below). We define by recursion a sequence of elements $(x_n, y_n) \in P^2$.

Let (x_0, y_0) be a pair such that x_0 and y_0 are incompatible in P and $P_{\succeq x_0}$ contains arbitrarily large finite strong antichains. Suppose we have defined x_n and y_n. Using arithmetical comprehension, search for a pair (x_{n+1}, y_{n+1}) such that $x_n \preceq x_{n+1}, y_{n+1}$, x_{n+1} and y_{n+1} are incompatible in P, and $P_{\succeq x_{n+1}}$ contains arbitrarily large finite strong antichains.

To show that the recursion never stops assume that $U \subseteq P$ is a final interval with arbitrarily large finite strong antichains ($U = P$ at stage 0, $U = P_{\succeq x_n}$ at stage $n + 1$). By Lemma 4.3 there exists a maximal strong antichain $S \subseteq U$ with at least two elements. By hypothesis, S is finite and we apply the following claim:

Claim. *There exists $x \in S$ such that $P_{\succeq x}$ contains arbitrarily large finite strong antichains.*

Proof of claim. Let $n = |S|$. We first show that for every $k \geq 1$ there exists $u \in S$ such that $P_{\succeq u}$ contains a strong antichain of size k.

Given $k \geq 1$, let T be a strong antichain of size $n \cdot k$. Since S is maximal, every element $y \in T$ is compatible with some element of S. For any $y \in T$ let $(u(y), v(y))$ be the least pair such that $u(y) \in S$ and $u(y), y \preceq v(y)$. Then $\{v(y) \colon y \in T\}$ is again a strong antichain of size $n \cdot k$. As $y \mapsto u(y)$ defines a function from T to S, it easily follows that for some $u \in S$ the upper cone $P_{\succeq u}$ contains at least k elements of the form $v(y)$ with $y \in T$.

Now, for all $k \geq 1$, let $u_k \in S$ be such that $P_{\succeq u_k}$ contains a strong antichain of size k. Since S is finite, by the infinite pigeonhole principle (which is provable in ACA_0), there exists $x \in S$ such that $x = u_k$ for infinitely many k. The upper cone $P_{\succeq x}$ thus contains arbitrarily large finite strong antichains. $\square$

In particular, $x_n \preceq y_m$ for all $n < m$ and x_n and y_n are incompatible in P. It follows that y_n is incompatible with y_m for all $n < m$. Then $\{y_n \colon n \in \mathbb{N}\}$ is an infinite strong antichain, for the desired contradiction. $\square$

The following Theorem shows that our use of ACA_0 in several of the preceding Lemmas is necessary and establish the reverse mathematics

results about Theorem 1.2 and the left to right directions of Theorems 1.1 and 1.3 (these are respectively conditions (3), (5), and (4) in the statement of the Theorem). We also show that apparently weaker statements, such as the restriction of Theorems 1.1 and 1.3 to well-partial orders, require ACA_0.

Theorem 4.5. *Over RCA_0, the following are pairwise equivalent:*

(1) ACA_0;

(2) *every partial order with no arbitrarily large finite strong antichains is a finite union of ideals;*

(3) *every partial order with no infinite strong antichains does not contain arbitrarily large finite strong antichains;*

(4) *every partial order with no infinite strong antichains is a finite union of ideals;*

(5) *if a partial order has no infinite antichains then every initial interval is a finite union of ideals;*

(6) *every well-partial order is a finite union of ideals.*

Proof. (1) $\implies$ (2) is Lemma 4.2 and (1) $\implies$ (3) is Lemma 4.4. The combination of Lemma 4.4 and Lemma 4.2 shows (1) $\implies$ (4). Since a strong antichain in a subset of a partial order is an antichain, (4) $\implies$ (5) holds. For (5) $\implies$ (6), recall that, provably in RCA_0, a well-partial order has no infinite antichains.

It remains to show that each of (2), (3) and (6) implies ACA_0. Reasoning in RCA_0 fix a one-to-one function $f\colon \mathbb{N} \to \mathbb{N}$. In each case we build a suitable partial order P which encodes the range of f.

We start with (2) $\implies$ (1). Let $P = \{a_n, b_n \colon n \in \mathbb{N}\} \cup \{c\}$. We define a partial order on P by letting:

(i) $a_n \preceq c$ for all n;

(ii) $b_n \preceq b_m$ for $n \leq m$;

(iii) $a_n \preceq b_m$ if and only if $(\exists i < m)f(i) = n$;

and adding no other comparabilities. It is easy to verify that every strong antichain in P has at most 2 elements. By (2) P is a finite union of ideals $A_0, \ldots, A_k$. By Lemma 3.3, we may assume that this union is essential. Let us assume $b_0 \in A_0$.

By $\mathbf{\Sigma}_1^0$-induction (actually $\mathbf{\Sigma}_0^0$) we prove that $(\forall m)(b_m \in A_0)$. The base step is obviously true. Suppose $b_m \in A_0$ and $b_{m+1} \notin A_0$. Then $A_0 = \{x \in P \colon x \preceq b_m\}$ (because every element $\succ b_m$ is $\succeq b_{m+1}$). Suppose $b_{m+1} \in A_1$. Then $A_0 \subseteq A_1$ and the decomposition is not essential, a contradiction. Therefore, A_0 contains all the b_m's. Now, it is straightforward to see that $(\exists m)f(m) = n$ if and only if $a_n \in A_0$, so that the range of f can be defined by $\mathbf{\Delta}_0^0$ comprehension.

To prove (3) $\implies$ (1) we exploit the notion of false and true stage. Recall that $n \in \mathbb{N}$ is said to be a *false stage for f* (or simply *false*) if $f(k) < f(n)$ for some $k > n$ and *true* otherwise. We may assume

to have infinitely many false stages, since otherwise the range of f exists by $\mathbf{\Delta}^0_1$ comprehension. On the other hand, there are always infinitely many true stages (i.e. for every m there exists $n > m$ which is true), because otherwise we can build an infinite descending sequence of natural numbers.

Let $P = \{a_n, b_n : n \in \mathbb{N}\}$ and define

  (i) $b_n \preceq b_m$ for all $n < m$;
  (ii) $a_n \preceq b_m$ if and only if $f(k) < f(n)$ for some k with $n < k \leq m$ (i.e. if at stage m we know that n is false);

and there are no other comparabilities.

Notice that the b_n's and the a_n's with n false are pairwise compatible in P. Therefore every infinite strong antichain in P consists of infinitely many a_n's with n true and at most one b_n or a_n with n false. Possibly removing that single element we have an infinite set of true stages. From this in RCA_0 we can obtain a strictly increasing enumeration of true stages $i \mapsto n_i$. Since $(\exists n)f(n) = m$ if and only if $(\exists n \leq n_m)f(n) = m$, the range of f exists by $\mathbf{\Delta}^0_1$ comprehension. Thus the existence of an infinite strong antichain in P implies the existence of the range of f in RCA_0.

To apply (3) and conclude the proof we need to show that P contains arbitrarily large finite strong antichains. To do this apparently we need $\mathbf{\Sigma}^0_2$-induction (which is not available in RCA_0) to show that for all k there exists k distinct true stages.

To remedy this problem (with the same trick used for this purpose in [MS11, Lemma 4.2]) we replace each a_n with $n+1$ distinct elements. Thus we set $P' = \{a_n^i, b_n : n \in \mathbb{N}, i \leq n\}$ and substitute (ii) with $a_n^i \leq_{P'} b_m$ if and only if $f(k) < f(n)$ for some k with $n < k \leq m$. Then also the existence of an infinite strong antichain in P' suffices to define the range of f in RCA_0. However the existence of arbitrarily large finite strong antichains in P' of the form $\{a_n^i : i \leq n\}$ follows immediately from the existence of infinitely many true stages.

We now show (6) $\implies$ (1). We again use false and true stages and as before we assume to have infinitely many false stages. The idea for P is to combine a linear order $P_0 = \{a_n : n \in N\}$ of order type $\omega + \omega^*$ with a linear order $P_1 = \{b_n : n \in \mathbb{N}\}$ of order type ω. The false and true stages give rise respectively to the ω and ω^* part of P_0, and every false stage is below some element of P_1. We proceed as follows.

Let $P = \{a_n, b_n : n \in \mathbb{N}\}$. For $n \leq m$, set

  (i) $a_n \preceq a_m$ if $f(k) < f(n)$ for some $n < k \leq m$ (i.e. if at stage m we know that n is false);
  (ii) $a_m \preceq a_n$ if $f(k) > f(n)$ for all $n < k \leq m$ (i.e. if at stage m we believe n to be true).

When condition (i) holds, we also put $a_n \preceq b_m$. Then we linearly order the b_m's by putting $b_i \preceq b_j$ if and only if $i \leq j$. There are no other comparabilities.

It is not difficult to verify that P is a partial order with no infinite antichains. Note that if n is false and $m > n$ is such that $f(m) < f(n)$, then $\{i \colon a_i \preceq a_n\} \subseteq \{i \colon i < m\}$ is finite, while if n is true, then $\{i \colon a_n \preceq a_i\} \subseteq \{i \colon i \leq n\}$ is finite. This explains our assertion that P_0 has order type $\omega + \omega^*$.

First assume that P is not a well-partial order. By definition, there exists $g \colon \mathbb{N} \to P$ such that $i < j$ implies $g(i) \npreceq g(j)$. As for every false n there are only finitely many $x \in P$ such that $a_n \npreceq x$, we must have $g(i) \neq a_n$ for all i and for all false n. We may assume that $g(i) \neq b_n$ for all i, n, since there are finitely many b_m such that $b_n \npreceq b_m$. We thus have $g(i) = a_{n_i}$ with n_i true for all i. Since $a_m \succ a_n$ and $n < m$ imply n false, the map $i \mapsto n_i$ is a strictly increasing enumeration of true stages. As before, the range of f exists by $\mathbf{\Delta}_1^0$ comprehension.

We now assume that P is a well-partial order. Apply (6), so that $P = \bigcup \{A_i \colon i < k\}$ is a finite union of ideals. By Lemma 3.3 we may assume that the union is essential so that there exists an ideal, say A_0, that contains all the b_m's.

We claim that n is false if and only if $a_n \in A_0$. To see this, let n be false. Thus $a_n \preceq b_m$ for some m, and hence $a_n \in A_0$. Conversely, if $a_n \in A_0$ then it is compatible with, for instance, b_0, and yet again it is $\preceq b_m$ for some m. Hence, the set of true stages is $\{n \colon a_n \notin A_0\}$, and the conclusion follows as before. □

4.2. **Equivalences with** ATR_0. We now consider the left to right direction of Theorem 1.5, i.e. the statement every countable scattered partial order with no infinite antichains has countably many initial intervals. We start with a technical Lemma:

Lemma 4.6 (ACA_0). *If a partial order P has perfectly many initial intervals, then there exists $x \in P$ such that either*

> *(i) $P_{\perp x}$ has uncountably many initial intervals, or*
> *(ii) both $P_{\prec x}$ and $P_{\succ x}$ have uncountably many initial intervals.*

Proof. Let P be a partial order with perfectly many initial intervals. Let $T \subseteq T(P)$ be a perfect tree.

We first show that there exist $x \in P$ such that both

$$\{I \in \mathcal{I}(P) \colon x \notin I\} \text{ and } \{I \in \mathcal{I}(P) \colon x \in I\}$$

are uncountable. Let $\tau \in T$ be such that both $\tau_0 = \tau^\smallfrown \langle 0 \rangle$ and $\tau_1 = \tau^\smallfrown \langle 1 \rangle$ belong to T. Let $x = |\tau|$ and notice that $x \in P$. For $i < 2$ define $T_i = \{\sigma \in T \colon \sigma \sqsubseteq \tau_i \vee \tau_i \sqsubseteq \sigma\}$. The trees T_0 and T_1 are perfect and witness the fact that the two collections of initial intervals are uncountable.

Now, suppose that condition (i) fails and let $\mathcal{I}(P_{\perp x}) = \{J_n \colon n \in \mathbb{N}\}$. We aim to show that (ii) holds.

Suppose for a contradiction that $P_{\prec x}$ has countably many initial intervals and let $\mathcal{I}(P_{\prec x}) = \{I_n \colon n \in \mathbb{N}\}$. Then it is not difficult to show that

$$\{I \in \mathcal{I}(P) \colon x \notin I\} = \{I_n \cup \downarrow J_m \colon n, m \in \mathbb{N}\}.$$

This contradicts the fact that $\{I \in \mathcal{I}(P) \colon x \notin I\}$ is uncountable.

Similarly, suppose that $P_{\succ x}$ has countably many initial intervals and let $\mathcal{I}(P_{\succ x}) = \{I_n \colon n \in \mathbb{N}\}$. Then, it is not difficult to show that

$$\{I \in \mathcal{I}(P) \colon x \in I\} = \{\downarrow(\{x\} \cup I_n \cup J_m) \colon n, m \in \mathbb{N}\}.$$

This contradicts the fact that $\{I \in \mathcal{I}(P) \colon x \in I\}$ is uncountable. Therefore, condition (ii) holds. $\qquad\square$

Theorem 4.7 (ATR$_0$). *Every scattered partial order with no infinite antichains has countably many initial intervals.*

Proof. Let P be a partial order with uncountably many initial intervals.

Let $\mathrm{Fin}(P)$ the set of (codes of) finite subsets of P. For all $F, G, H \in \mathrm{Fin}(P)$, let

$$P_{F,G,H} = \bigcap_{x \in F} P_{\prec x} \cap \bigcap_{x \in G} P_{\succ x} \cap \bigcap_{x \in H} P_{\perp x}.$$

We want to define a pruned tree $T \subseteq 3^{<\mathbb{N}}$ and a function $f \colon T \to \mathrm{Fin}(P)^3$ such that the following hold (where $f(\sigma) = (F_\sigma, G_\sigma, H_\sigma)$ and $P_\sigma = P_{f(\sigma)}$):

(i) $f(\langle\rangle) = (\emptyset, \emptyset, \emptyset)$;
(ii) for all $\sigma \in T$, $\sigma^\frown\langle 0\rangle \in T$ if and only if $\sigma^\frown\langle 1\rangle \in T$ if and only if $\sigma^\frown\langle 2\rangle \notin T$ (in other words there are two possibilities: either exactly $\sigma^\frown\langle 0\rangle$ and $\sigma^\frown\langle 1\rangle$ belong to T, or only $\sigma^\frown\langle 2\rangle \in T$);
(iii) if $\sigma^\frown\langle 0\rangle \in T$, then $f(\sigma^\frown\langle 0\rangle) = (F_\sigma \cup \{x\}, G_\sigma, H_\sigma)$ and $f(\sigma^\frown\langle 1\rangle) = (F_\sigma, G_\sigma \cup \{x\}, H_\sigma)$ for some $x \in P_\sigma$;
(iv) if $\sigma^\frown\langle 2\rangle \in T$, then $f(\sigma^\frown\langle 2\rangle) = (F_\sigma, G_\sigma, H_\sigma \cup \{x\})$ for some $x \in P_\sigma$.

We first show that if there exist T and f as above, then P is not scattered or it contains an infinite antichain.

First suppose there exists a path $g \in [T]$ such that $g(n) = 2$ for infinitely many n. Then let

$$D = \bigcup_{n \in \mathbb{N}} H_{g \restriction n}.$$

It is easy to check, using (iv) and the definition of $P_{F,G,H}$, that D is an infinite antichain.

If there are no paths $g \in [T]$ such that $g(n) = 2$ for infinitely many n then it is easy to see, using (ii), that T is perfect. For all $\sigma^\frown\langle 0\rangle \in T$,

let x_σ be the unique element of $F_{\sigma^\frown\langle 0\rangle} \setminus F_\sigma$. We claim that

$$Q = \{x_\sigma : \sigma^\frown\langle 0\rangle \in T\}$$

is a dense linear order in P.

We first note that $x_\sigma \neq x_\tau$ for $\sigma, \tau \in T$ with $\sigma \neq \tau$. Now fix distinct $x_\sigma, x_\tau \in Q$ with the goal of showing that they are comparable in P and that there exists an element of Q strictly between them. First assume that σ and τ are comparable as sequences, let us say $\sigma \sqsubset \tau$. Then, using (iii), $x_\tau \prec x_\sigma$ if $\sigma^\frown\langle 0\rangle \sqsubseteq \tau$ and $x_\sigma \prec x_\tau$ if $\sigma^\frown\langle 1\rangle \sqsubseteq \tau$. Suppose $x_\tau \prec x_\sigma$ (the other case is similar) and let $\eta \in T$ so that $\tau^\frown\langle 1\rangle \sqsubseteq \eta$ and $\eta^\frown\langle 0\rangle \in T$. Then $x_\tau \prec x_\eta \prec x_\sigma$ by (iii). Suppose now that σ and τ are not one initial segment of the other. We may assume that $\eta^\frown\langle 0\rangle \sqsubseteq \sigma$ and $\eta^\frown\langle 1\rangle \sqsubseteq \tau$ for some η. Then $x_\eta \in Q$ and, using (iii) again, $x_\sigma \prec x_\eta \prec x_\tau$.

It remains to show that we can define T and f satisfying (i)–(iv).

By Theorem 2.5, P has perfectly many initial intervals. Let U be a perfect subtree of $T(P)$. By Corollary 2.8, there exists a countable coded ω-model M such that $P, U \in M$ and M satisfies Σ_1^1-DC_0 and ATR_0^P.

We recursively define T and f using M as a parameter. Let $\langle\rangle \in T$ and $f(\langle\rangle) = (\emptyset, \emptyset, \emptyset)$ as required by (i). Note that M satisfies "$T(P_{\langle\rangle})$ contains a perfect subtree". Let $\sigma \in T$ and assume by arithmetical induction that M satisfies "$T(P_\sigma)$ contains a perfect subtree". Since M is a model of ACA_0, by Lemma 4.6 applied to P_σ, there exists $x \in P_\sigma$ such that either

- (a) M satisfies "$T(P_\sigma \cap P_{\perp x})$ has uncountably many paths", or
- (b) M satisfies "both $T(P_\sigma \cap P_{\prec x})$ and $T(P_\sigma \cap P_{\succ x})$ have uncountably many paths".

Search for the least x with this arithmetical property. If (a) holds (and we can check this arithmetically outside M), use ATR_0^P within M to apply Theorem 2.6 to the P-computable tree $T(P_\sigma \cap P_{\perp x})$. We obtain that M satisfies "$T(P_\sigma \cap P_{\perp x})$ contains a perfect subtree". Thus, let $\sigma^\frown\langle 2\rangle \in T$ and set $f(\sigma^\frown\langle 2\rangle) = (F_\sigma, G_\sigma, H_\sigma \cup \{x\})$. If (b) holds, then arguing analogously we obtain that M satisfies "both $T(P_\sigma \cap P_{\prec x})$ and $T(P_\sigma \cap P_{\succ x})$ contain perfect subtrees". Thus let $\sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in T$ and set

$$f(\sigma^\frown\langle 0\rangle) = (F_\sigma \cup \{x\}, G_\sigma, H_\sigma) \text{ and } f(\sigma^\frown\langle 1\rangle) = (F_\sigma, G_\sigma \cup \{x\}, H_\sigma).$$

In any case, (ii)-(iv) are satisfied and the induction hypothesis that M satisfies "$T(P_\sigma)$ contains a perfect subtree" is preserved. $\square$

Theorem 4.8. *Over ACA_0, the following are equivalent:*

- *(1) ATR_0;*
- *(2) every scattered partial order with no infinite antichains has countably many initial intervals;*

(3) every scattered linear order has countably many initial intervals.

Proof. (1) $\implies$ (2) is Theorem 4.7 and (2) $\implies$ (3) is immediate. We show (3) $\implies$ (1) by essentially repeating the proof of [Clo89, Theorem 18].

Assume ACA_0. We wish to prove ATR_0. By [Sim09, Theorem V.5.2], ATR_0 is equivalent (over RCA_0) to the statement asserting that for every sequence of trees $\{T_i \colon i \in \mathbb{N}\}$ such that every T_i has at most one path, there exists the set $\{i \in \mathbb{N} \colon [T_i] \neq \emptyset\}$. So let $\{T_i \colon i \in \mathbb{N}\}$ be such a sequence. Let us order each T_i with the Kleene-Brouwer order $\leq_{\mathrm{KB}}$ and define the linear order $L = \sum_{i \in \mathbb{N}} T_i$

We aim to show that L is scattered. By Lemma 2.3, it suffices to prove that every T_i is scattered. To this end, we show that if a tree T has at most one path then the Kleene-Brouwer order on T is of the form

$$(*) \qquad\qquad X + \sum_{n \in \omega^*} Y_n,$$

where X and the Y_n are (possibly empty) well-orders. Applying Lemma 2.3 again, we obtain that T is scattered.

If T has no path, then ACA_0 proves that $\leq_{\mathrm{KB}}$ well-orders T, and hence we can take $X = T$ and the Y_n's empty. Now let f be the unique path of T. Let $X = \{\sigma \in T \colon (\forall n)\sigma <_{\mathrm{KB}} f \restriction n\}$ and $Y_n = \{\sigma \in T \colon f \restriction n+1 <_{\mathrm{KB}} \sigma \leq_{\mathrm{KB}} f \restriction n\}$, for all $n \in \mathbb{N}$. It is straightforward to see that $(*)$ holds. We now claim that X is a well-order. Suppose not, and let $(\sigma_n)_{n \in \mathbb{N}}$ be an infinite descending sequence in X. Form the tree $T_0 = \{\sigma \in T \colon (\exists n)\sigma \sqsubseteq \sigma_n\}$. Then T_0 is not well-founded and so it has a path. As T_0 is a subtree of T, this path must be f. Let $i \in \mathbb{N}$ be such that $\sigma_0 \restriction i = f \restriction i$ and $\sigma_0(i) < f(i)$ (such an i exists because $\sigma_0 \in X$). On the other hand, $f \restriction i+1 \in T_0$, and thus $f \restriction i+1 \sqsubseteq \sigma_n$ for some $n \in \mathbb{N}$. It follows that $\sigma_0 <_{\mathrm{KB}} \sigma_n$, a contradiction. To show that each Y_n is a well-order notice that $Y_n = \{\sigma \in T \colon f \restriction n \sqsubset \sigma \wedge f(n) < \sigma(n)\} \cup \{f \restriction n\}$.

Apply (3) to L and let $\mathcal{I}(L) = \{I_n \colon n \in \mathbb{N}\}$. It is easy to check that T_i has a path if and only if

$$(\exists n)\Big(\bigcup_{j<i} T_j \subseteq I_n \wedge T_i \not\subseteq I_n \wedge L \setminus I_n \text{ has no least element}\Big).$$

Therefore, the set $\{i \in \mathbb{N} \colon [T_i] \neq \emptyset\}$ exists by arithmetical comprehension. $\qquad\square$

It is worth noticing that a natural weakening of condition (3) of Theorem 4.8 is provable in RCA_0:

Lemma 4.9 (RCA_0)**.** *Every linear order with perfectly many initial intervals is not scattered.*

Proof. Let L be a linear order and $T \subseteq T(L)$ be a perfect tree. Define

$$Q = \{x \in L \colon (\exists \sigma \in T)(|\sigma| = x \wedge \sigma^\frown\langle 0\rangle, \sigma^\frown\langle 1\rangle \in T)\}.$$

The argument showing that Q is a dense subset of L is similar to the one in the proof of Theorem 4.7. $\square$

5. The right to left directions

In this section we study the right to left directions of Theorems 1.1, 1.3, and 1.5. The right to left direction of Theorem 1.5 naturally splits into two statements with the same hypothesis (the existence of countably many initial intervals) and different conclusions (the partial order is scattered and the partial order has no infinite antichains). We have thus four different statements altogether. All these statements have simple proofs in ACA_0, but it turns out that each of them can be proved in a properly weaker system.

5.1. **Proofs in RCA_0.** We start with a simple observation about the right to left direction of Theorem 1.3.

Lemma 5.1 (RCA_0). *Every partial order which is a finite union of ideals has no infinite strong antichains.*

Proof. Since an ideal does not contain incompatible elements, if the partial order is the union of k ideals we have even a finite bound on the size of strong antichains. $\square$

Another statement that can be proved in RCA_0 is the following half of the right to left direction of Theorem 1.5.

Theorem 5.2 (RCA_0). *Every partial order with countably many initial intervals is scattered.*

Proof. We show that if P is not scattered, then P has perfectly many initial intervals. By Lemma 2.2 we may assume that P contains a dense linear order Q.

We define by recursion an embedding $f\colon 2^{<\mathbb{N}} \to T(P)$. Thus $T_0 = \{\tau \in T(P)\colon (\exists \sigma \in 2^{<\mathbb{N}})\tau \sqsubseteq f(\sigma)\}$ is a perfect subtree of $T(P)$. Since $\tau \in T_0$ if and only if $(\exists \sigma \in 2^{<\mathbb{N}})(|\sigma| = |\tau| \wedge \tau \sqsubseteq f(\sigma))$, T_0 exists in RCA_0.

We say that $x \in P$ is *free for* $\tau \in T(P)$ if

$$(\forall y < |\tau|)((\tau(y) = 1 \implies x \not\preceq y) \wedge (\tau(y) = 0 \implies y \not\preceq x)).$$

In other words, x is free for τ if and only if there exist $\tau_0, \tau_1 \in T(P)$ with $\tau \sqsubset \tau_i$ and $\tau_i(x) = i$. Since $T(P)$ is a pruned tree this means that there exist two initial intervals of P whose characteristic function extends τ, one containing x and the other avoiding x.

Let $f(\langle\rangle) = \langle\rangle$. Suppose we have defined $f(\sigma) = \tau$. Assume by Σ_1^0 induction that Q contains at least two (and hence infinitely many) elements that are free for τ. Then search for $a \prec b \prec c$ in Q that are free for τ. We will define $\tau_0, \tau_1 \in T(P)$ which are extensions of τ with

$|\tau_i| = b + 1$ and $\tau_i(b) = i$. Thus τ_0 and τ_1 are incompatible and we can let $f(\sigma^\frown \langle i \rangle) = \tau_i$.

We show how to define τ_0 (to define τ_1 replace a with b and b with c). Since $\{x \in P \colon x < b\}$ is finite, we can find $a', b' \in Q$ with $a \prec a' \prec b' \prec b$ such that $a', b' > b$, and for no $x \in P$ with $x < b$ we have $a' \prec x \prec b'$. Given $x < |\tau_0|$ we need to define $\tau_0(x)$, and we proceed by cases (notice that the first three conditions are determined by the fact that we want $\tau_0 \in T(P)$ and $\tau_0 \sqsupseteq \tau$):

- if $x \notin P$ let $\tau_0(x) = 0$;
- if $x \in P$ is not free for τ because there exists $y < |\tau|$ such that $\tau(y) = 0$ and $y \preceq x$ let $\tau_0(x) = 0$;
- if $x \in P$ is not free for τ because there exists $y < |\tau|$ such that $\tau(y) = 1$ and $x \preceq y$ let $\tau_0(x) = 1$;
- if x is free for τ we define $\tau_0(x)$ according to the following cases:
    - (i) if $x \prec a'$, let $\tau_0(x) = 1$;
    - (ii) if $x \succ b'$, let $\tau_0(x) = 0$;
    - (iii) otherwise, let $\tau_0(x) = 0$.

It is not difficult to check that τ_0 extends τ, $\tau_0(b) = 0$ and both a' and b' are free for τ_0, preserving the induction hypothesis. $\square$

With regard to the other half of the right to left direction of Theorem 1.5, RCA_0 proves the following.

Lemma 5.3 (RCA_0). *An infinite antichain has perfectly many initial intervals.*

Proof. If P is an antichain then the tree $T(P)$ consists of all $\sigma \in 2^{<\mathbb{N}}$ such that $x \notin P$ implies $\sigma(x) = 0$. If P is infinite it is immediate that this tree is perfect and thus Lemma 2.4 implies that P has perfectly many initial intervals. $\square$

5.2. **Proofs in** WKL_0. We now look at the right to left direction of Theorem 1.1, which states that every partial order with an infinite antichain contains an initial interval that cannot be written as a finite union of ideals. The proof can be carried out very easily in ACA_0: just take the downward closure of the given antichain. We improve this upper bound by showing that WKL_0 suffices. We first point out that RCA_0 proves a particular instance of the statement.

Lemma 5.4 (RCA_0). *Let P be a partial order with a maximal (with respect to inclusion) infinite antichain. Then there exists an initial interval that is not a finite union of ideals.*

Proof. Let D be a maximal infinite antichain of P. The maximality of D implies that for all $x \in P$ we have

$$(\exists d \in D)x \preceq d \iff \neg(\exists d \in D)d \prec x.$$

Therefore the downward closure of D is $\mathbf{\Delta}_1^0$ definable and thus exists in RCA_0. Letting $I = \{x \in P \colon (\exists d \in D)x \preceq d\}$ we obtain an initial interval which is not a finite union of ideals, since distinct elements of D are incompatible in I. $\qquad\square$

To use Lemma 5.4 in the general case we need to extend an infinite antichain to a maximal one. It is easy to show that RCA_0 proves the existence of maximal antichains in any partial order. On the other hand, we now show in RCA_0 that the statement that every antichain is contained in a maximal antichain is equivalent to ACA_0.

Lemma 5.5. *Over* RCA_0, *the following are equivalent:*

(1) ACA_0;
(2) *every antichain in a partial order extends to a maximal antichain.*

Proof. We first show (1) $\implies$ (2). Let P be a partial order and $D \subseteq P$ be an antichain. By recursion we define $f \colon \mathbb{N} \to \{0,1\}$ by letting $f(x) = 1$ if and only if $D \cup \{y < x \colon f(y) = 1\} \cup \{x\}$ is an antichain in P. Then $E = \{x \colon f(x) = 1\}$ is a maximal antichain with $D \subseteq E$.

For the reversal argue in RCA_0 and fix a one-to-one function $f \colon \mathbb{N} \to \mathbb{N}$. Let $P = \{a_n, b_n \colon n \in \mathbb{N}\}$ and define the partial order by letting $b_m \preceq a_n$ if and only if $f(m) = n$, and adding no other comparabilities. Then apply (2) to the antichain $D = \{b_m \colon m \in \mathbb{N}\}$ and obtain a maximal antichain E such that $D \subseteq E$. It is immediate that $(\exists m)f(m) = n$ if and only if $a_n \notin E$, so that in RCA_0 we can prove the existence of the range of f. $\qquad\square$

We now show how to prove the right to left direction of Theorem 1.1 in WKL_0.

Theorem 5.6 (WKL_0)**.** *Every partial order with an infinite antichain contains an initial interval that is not a finite union of ideals.*

Proof. Let P be a partial order containing an infinite antichain D. Let $\varphi(x)$ and $\psi(x)$ be the $\mathbf{\Sigma}_1^0$ formulas $x \in D$ and $(\exists y)(y \in D \wedge y \prec x)$ respectively. It is obvious that $(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \npreceq x)$. By $\mathbf{\Sigma}_1^0$ initial interval separation (Lemma 3.1), there exists an initial interval $I \subseteq P$ such that

$$(\forall x \in P)((\varphi(x) \implies x \in I) \wedge (\psi(x) \implies x \notin I)).$$

Therefore, I contains D and no element above any element of D. To see that I cannot be the union of finitely many ideals notice that distinct $x, x' \in D$ cannot belong to the same ideal $A \subseteq I$, for otherwise there would be $z \in I$ such that $x, x' \preceq z$, which implies $\psi(z)$. $\qquad\square$

We do not know whether the statement of Theorem 5.6 implies WKL_0. Notice however that the proof above uses the existence of an

initial interval I containing the infinite antichain D and no elements above any element of D. We now show that even the existence of an initial interval I containing infinitely many elements of the antichain D and no elements above any element of D is equivalent to $\mathsf{WKL_0}$. Therefore a proof of the right to left direction of Theorem 1.1 in a system weaker than $\mathsf{WKL_0}$ must avoid using such an I.

Lemma 5.7. *Over* $\mathsf{RCA_0}$, *the following are equivalent:*

  (1) $\mathsf{WKL_0}$;
  (2) if a partial order P contains an infinite antichain D, then P has an initial interval I such that $D \subseteq I$ and $(\forall x \in D)(\forall y \in I)x \not\succ y$;
  (3) if a partial order P contains an infinite antichain D, then P has an initial interval I such that $I \cap D$ is infinite and $(\forall x \in D)(\forall y \in I)x \not\succ y$.

Proof. The proof of $(1) \implies (2)$ is contained in Theorem 5.6 and $(2) \implies (3)$ is obvious, so that we just need to show $(3) \implies (1)$. Fix one-to-one functions $f, g \colon \mathbb{N} \to \mathbb{N}$ such that $(\forall n, m \in \mathbb{N})f(n) \neq g(m)$. Let $P = \{a_n, b_n \colon n \in \mathbb{N}\}$ the partial order defined by letting

  (i) $a_n \preceq b_m$ if $m = g(n)$;
  (ii) $b_k \preceq a_n$ if $(\exists i < n)(i < g(n) \wedge f(i) = k)$, i.e. k enters the range of f before stage $\min\{n, g(n)\}$;
  (iii) $b_k \preceq b_m$ if $(\exists i < m)(f(i) = k \wedge (\forall j < i)f(j) \neq m)$, i.e. k enters the range of f before stage m and when m has not entered the range of f yet,

and adding no other comparabilities.

To check that P is indeed a partial order we need to show that it is transitive. The main cases are the following:

  • If $b_k \preceq a_n \preceq b_m$ we have $m = g(n)$ and the existence of $i < \min\{n, m\}$ such that $f(i) = k$. By the hypothesis on f and g we have $f(j) \neq m$ for every j, and in particular for every $j < i$, so that $b_k \preceq b_m$ follows.
  • If $b_k \preceq b_m \preceq b_\ell$ there exist $i < m$ and $i' < \ell$ such that $f(i) = k$, $(\forall j < i)f(j) \neq m$, $f(i') = m$, and $(\forall j < i')f(j) \neq \ell$. The second and third condition imply $i \leq i'$, so that $i < \ell$, $(\forall j < i)f(j) \neq \ell$ and we obtain $b_k \preceq b_\ell$.
  • If $b_k \preceq b_m \preceq a_n$ there exist $i < m$ and $i' < n$ such that $f(i) = k$, $(\forall j < i)f(j) \neq m$, $i' < g(n)$, and $f(i') = m$. Again we obtain $i \leq i'$, so that $i < \min\{n, g(n)\}$ and we can conclude $b_k \preceq a_n$.

The set $D = \{a_n \colon n \in \mathbb{N}\}$ is an infinite antichain. Applying (3) we obtain an initial interval I of P which contains infinitely many elements of D and no elements above any element of D. We now check that $\{k \in \mathbb{N} \colon b_k \in I\}$ separates the range of f from the range of g.

If $k = g(n)$ it is immediate that $a_n \prec b_k$ so that $b_k \notin I$.

On the other hand suppose that $k = f(i)$. The set $A = \{n\colon g(n) \leq i\}$ is finite by the injectivity of g and we can let $m = \max(\{i\} \cup A)$. Since $D \cap I$ is infinite there exists $n > m$ such that $a_n \in I$. Then we have $i < n$ and $i < g(n)$ (because $n \notin A$), so that $b_k \preceq a_n$. Therefore $b_k \in I$. □

We notice that another weakening of statement (2) of Lemma 5.7 which is equivalent to WKL_0 is the following: "if a partial order P contains an infinite antichain D, then there exists an initial interval I such that $D \subseteq I$ and $(\forall y \in I)(\exists^\infty x \in D)x \not\prec y$" (the proof of the reversal uses the partial order of the proof above equipped with the inverse order). However this statement does not imply the statement of Theorem 5.6.

Our next goal is to show that WKL_0 suffices to prove the half of the right to left direction of Theorem 1.5 that was not proved in RCA_0 in Theorem 5.2. In other words, we study the statement that every partial order with countably many initial intervals has no infinite antichains. Understanding initial intervals of partial orders with an infinite antichain leads to study the relationship between initial intervals of partial orders contained one into the other.

Lemma 5.8. *Over* RCA_0, *the following are equivalent:*

(1) WKL_0;

(2) Let Q and P be partial orders and f be an embedding of Q into P. Then

$$\mathcal{I}(Q) = \{f^{-1}(J)\colon J \in \mathcal{I}(P)\};$$

(3) Let Q be a subset of a partial order P. Then $\mathcal{I}(Q) = \{J \cap Q\colon J \in \mathcal{I}(P)\}$.

Proof. We start with (1) $\implies$ (2). Let $f\colon Q \to P$ be an embedding. It is easy to check that if $J \in \mathcal{I}(P)$ then $f^{-1}(J) \in \mathcal{I}(Q)$, so that the right to left inclusion is established even in RCA_0.

For the other inclusion fix $I \in \mathcal{I}(Q)$. Let $\varphi(x)$ and $\psi(x)$ be the Σ_1^0 formulas $(\exists y \in Q)(y \in I \wedge x = f(y))$ and $(\exists y \in Q)(y \notin I \wedge x = f(y))$ respectively. Since f is an embedding and I is an initial interval, we have

$$(\forall x, y \in P)(\varphi(x) \wedge \psi(y) \implies y \not\preceq_P x).$$

Apply Σ_1^0 initial interval separation (Lemma 3.1) to get $J \in \mathcal{I}(P)$ such that $f(I) \subseteq J$ and $J \cap f(Q \setminus I) = \emptyset$. It is immediate that $I = f^{-1}(J)$.

Since the implication (2) $\implies$ (3) is obvious, it remains to show (3) $\implies$ (1).

Instead of WKL_0, we prove statement (3) of Lemma 3.1, i.e. initial interval separation. Let P be a partial order and $A, B \subseteq P$ such that $(\forall x \in A)(\forall y \in B)y \not\preceq x$. Let $Q = A \cup B \subseteq P$ and notice that

$A \in \mathcal{I}(Q)$. By (3) there exists $J \in \mathcal{I}(P)$ such that $A = J \cap Q$. It is easy to see that $A \subseteq J$ and $J \cap B = \emptyset$, completing the proof.   □

Notice that the obvious proof of the nontrivial direction of (2), namely given $I \in \mathcal{I}(Q)$ let J be the downward closure of $f(I)$, uses arithmetical comprehension.

Corollary 5.9 (WKL_0). *Let P and Q be partial orders such that Q embeds into P. If P has countably many initial intervals, then Q does.*

Proof. Fix an embedding $f\colon Q \to P$. Let $\{J_n\colon n \in \mathbb{N}\}$ be such that for all $J \in \mathcal{I}(P)$ there exists n with $J = J_n$. For every n let $I_n = f^{-1}(J_n)$, which exists in RCA_0. Then, by Lemma 5.8, for all $I \in \mathcal{I}(Q)$ there exists n with $I = I_n$, showing that Q has countably many initial intervals.   □

We can now prove in WKL_0 the part of the right to left direction of Theorem 1.5 we are interested in.

Theorem 5.10 (WKL_0). *Every partial order with countably many initial intervals has no infinite antichains.*

Proof. Immediate from Lemma 5.3 and Corollary 5.9.   □

5.3. **Unprovability in RCA_0.** In this subsection we show that RCA_0 does not suffice to prove the statements shown in Theorems 5.6 and 5.10 to be provable in WKL_0.
    A single construction actually works for both statements.

Lemma 5.11. *There exists a computable partial order P with an infinite computable antichain such that any computable initial interval of P is the downward closure of a finite subset of P.*

Before proving Lemma 5.11 we show how to deduce from it the unprovability results.

Theorem 5.12. RCA_0 *does not prove that every partial order such that all its initial intervals are finite union of ideals has no infinite antichains.*

Proof. It suffices to show that the statement fails in **REC**, the ω-model of computable sets. Let P the computable partial order of Lemma 5.11 and let I be a computable initial interval of P. Let F be a finite set such that $I = \downarrow F$. Then $I = \bigcup_{x \in F} P_{\preceq x}$ and each $P_{\preceq x}$ is a computable ideal.
    Thus all initial intervals of P which belong to **REC** are finite union of ideals also belonging to **REC**. On the other hand, P has an infinite antichain in **REC**, showing the failure of the statement.   □

Theorem 5.13. RCA_0 *does not prove that every partial order with countably many initial intervals has no infinite antichains.*

Proof. We again show that the statement fails in **REC**, and again use the computable partial order P of Lemma 5.11. Since the downward closures of finite subsets of P are uniformly computable, there exists a set $\{I_n \colon n \in N\}$ in **REC** which lists all computable initial intervals of P. Therefore **REC** satisfies that P has countably many initial intervals. Since P has an infinite antichain in **REC**, the statement fails. $\square$

Proof of Lemma 5.11. We build P by a finite injury priority argument. We let $P = \{x_n, y_n \colon n \in \omega\}$ and ensure the existence of an infinite computable antichain by making the x_n's pairwise incomparable.

We further make sure that, for all $e \in \omega$, P meets the requirement:

$$R_e \colon (\exists y)\big((\Phi_e(y) = 1 \implies (\forall^\infty z \in P)z \preceq y) \wedge (\Phi_e(y) = 0 \implies (\forall^\infty z \in P)y \preceq z)\big).$$

Here, as usual, Φ_e is the function computed by the Turing machine of index e and $\forall^\infty$ means 'for all but finitely many'.

We first show that meeting all the requirements implies that P satisfies the statement of the Lemma. If I is a computable initial interval of P with characteristic function Φ_e, fix y given by R_e. We must have $\Phi_e(y) \in \{0, 1\}$. If $\Phi_e(y) = 0$ then, by R_e, $(\forall^\infty z \in P)y \preceq z$. As $y \notin I$, this implies that I is finite and hence $I = \downarrow I$ is the downward closure of a finite set. If $\Phi_e(y) = 1$, then by R_e we have $(\forall^\infty z \in P)z \preceq y$. Thus $P \setminus P_{\preceq y}$ and hence $I \setminus P_{\preceq y}$ are finite. As $y \in I$, $I = \downarrow(\{y\} \cup (I \setminus P_{\preceq y}))$ is the downward closure of a finite set.

Our strategy for meeting a single requirement R_e consists in fixing a witness y_n and waiting for a stage $s + 1$ such that

$$\Phi_{e,s}(y_n) \in \{0, 1\}.$$

If this never happens, R_e is satisfied. If $\Phi_{e,s}(y_n) = 0$, we put every x_m and y_m with $m > s$ above y_n. If $\Phi_{e,s}(y_n) = 1$, we put every x_m and y_m with $m > s$ below y_n. In this way R_e is obviously satisfied.

To meet all the requirements, the priority order is $R_0, R_1, R_2, \ldots$. At every stage s, we define a witness for R_e via an index $n_{e,s}$ and mark the requirements by a $\{0, 1\}$-valued function $r(e, s)$ such that $r(e, s) = 0$ if and only if R_e might require attention at stage s.

Stage s = 0. For all e, $n_{e,0} = e$ and $r(e, 0) = 0$.

Stage s + 1. We say that R_e *requires attention* at stage $s + 1$ if $e \leq s$, $n_{e,s} \leq s$, $r(e, s) = 0$ and $\Phi_{e,s}(y_{n_{e,s}}) \in \{0, 1\}$. If no R_e requires attention, then let $n_{i,s+1} = n_{i,s}$ and $r(i, s + 1) = r(i, s)$ for all i. Otherwise, let e be least such that R_e requires attention. Then R_e *receives attention* at stage $s + 1$ and $n = n_{e,s}$ is *activated* and declared *low* if $\Phi_{e,s}(y_n) = 0$, *high* if $\Phi_{e,s}(y_n) = 1$. Let $n_{e,s+1} = n_{e,s}$ and $r(e, s + 1) = 1$. For $i < e$, $n_{i,s+1} = n_{i,s}$ and $r(i, s + 1) = r(i, s)$. For $i > e$, $n_{i,s+1} = s + i - e$ and $r(i, s + 1) = 0$.

The following two properties are easily seen to hold:

(1) every n is activated at most once;
(2) if n is activated at stage s, then no m such that $n < m < s$ is activated after s.

We define $\preceq$ by stipulating that for all $n < m$:

(i) x_n is incomparable with each of y_n, x_m and y_m;
(ii) $y_n \preceq (\succeq) x_m, y_m$ if and only if n is activated at some stage s such that $n < s \leq m$, is declared low (high) and no $k < n$ is activated at any stage t such that $s < t \leq m$.

When (ii) occurs, it follows by (2) that no $k < n$ is activated at any stage t such that $n < t \leq m$.

Claim 1. *P is a partial order.*

Proof of claim. We use z_n to denote one of x_n and y_n.

To show antisymmetry, suppose for a contradiction that $z_n \preceq z_m$ and $z_m \preceq z_n$ with $n < m$. By (i) z_n must be y_n. Since n can be activated only once, it follows that n is activated at some stage s with $n < s \leq m$ and, by (ii), is declared both low and high, a contradiction.

To check transitivity, let $z_n \prec z_m \prec z_p$. Notice that n, m and p are all distinct. We consider the following cases:

(a) $n < m, p$. Then $z_n = y_n$ and n is activated and declared low at some stage s such that $n < s \leq m$. It is easy to verify that no $k < n$ is activated at any stage t such that $n < t \leq p$, and thus $y_n \preceq z_p$.
(b) $m < n, p$. Then $z_m = y_m$ and m is declared both high and low, contradiction.
(c) $p < n, m$. Then $z_p = y_p$ and p is activated and declared high at some stage s such that $p < s \leq m$. As in case (a), it is easy to check that no $k < p$ is activated at any stage t such that $p < t \leq n$, and so $z_n \preceq y_p$. $\square$

Claim 2. *Every R_e receives attention at most finitely often and is satisfied.*

Proof of claim. As usual, the proof is by induction on e. Let s be the least such that no R_i with $i < e$ receives attention after s. Let $n = n_{e,s}$. Then $n = n_{e,t}$ for all $t \geq s$, because a witness for a requirement changes only when a stronger priority requirement receives attention. Similarly, $r(e, t) = 0$ for all $t \geq s$ such that R_e has not received attention at any stage between s and t. If $\Phi_e(y_n) \notin \{0, 1\}$, R_e is clearly satisfied. Suppose that $\Phi_e(y_n) = 0$ (case 1 is similar) and let t be minimal such that $t \geq \max\{s, e, n\}$ and $\Phi_{e,t}(y_n) = 0$. Then R_e receives attention at stage $t + 1$, n is activated and declared low and no $m < n$ will be activated after stage $t + 1$ (because $n_{i,u} > n$ for all $i > e$ and $u > t$). Then $y_n \preceq x_m, y_m$ for all $m > t$ and so R_e is satisfied. $\square$

Claim 2 completes the proof of the Lemma. $\square$

6. Open problems

The results of Sections 5.2 and 5.3 leave open the status of the right to left directions of Theorems 1.1 and 1.5. Each of the statements (1) "every partial order with an infinite antichain contains an initial interval which is not a finite union of ideals" and (2) "every partial order with an infinite antichain has uncountably many initial intervals" can be either equivalent to WKL_0 or of strength strictly between RCA_0 and WKL_0.

The latter case would be quite interesting, since the only mathematical statements with this intermediate strength are those from measure theory that are equivalent to the system WWKL_0. Bienvenu, Patey, and Shafer improved Theorems 5.12 and 5.13 by showing that WWKL_0 does not imply neither (1) nor (2). These results are obtained by modifying the proof of Lemma 5.11. The draft [BPS] includes also other non-implications involving statements (1) (called NCF there) and (2).

On the other hand, Gregory Igusa (in private communications) claims that there cannot be a uniform proof of WKL_0 from (1). This claim does not rule out the possibility that (1) implies WKL_0: e.g. there might exist a proof using twice the statement, the second time using it on a partial order built from the initial interval obtained by the first application.

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Dipartimento di Matematica e Informatica, Università di Udine, 33100
Udine, Italy
  E-mail address: emanuele.frittaion@uniud.it

Dipartimento di Matematica e Informatica, Università di Udine, 33100
Udine, Italy
  E-mail address: alberto.marcone@uniud.it